Lecture VII.

$\Rightarrow \frac{\partial}{\partial t} W(\varphi_1, \cdots, \varphi_n) = \det\begin{pmatrix} \varphi_1 & \cdots, & \varphi_n \\ \vdots & & \vdots \\ \varphi_1^{(n-2)} & & \varphi_n^{(n-2)} \\ \varphi_1^{(n)}, & & \varphi_n^{(n)} \end{pmatrix} = \det\begin{pmatrix} \varphi_1 & \cdots, & \varphi_n \\ \vdots & & \vdots \\ \varphi_1^{(n-2)} & , & \varphi_n^{(n-2)} \\ -p_{n-1}\varphi_1^{(n-1)} \cdots - p_0\varphi_1, & & -p_{n-1}\varphi_n^{(n-1)} \cdots p_0\varphi_n \end{pmatrix} =$

$-p_{n-1} W(\varphi_1, \cdots, \varphi_n).$

//.

Thm 4.4: Suppose that the functions $p_0, \cdots, p_{n-1}$ are continuous on an open interval I. If $y = \varphi_1(t), \cdots, y = \varphi_n(t)$ are solutions to [c] and the Wronskian $W(\varphi_1, \cdots, \varphi_n)(t_0) \neq 0$, for some $t_0 \in I$, then every solution of [c] can be expressed as a linear combination of $\varphi_1, \cdots, \varphi_n$.

//.

pf: Let $y = \varphi(t)$ be a solution to [c], and suppose that $W(\varphi_1, \cdots, \varphi_n)(t_0) \neq 0$. Define $(y_0, y_1, \cdots, y_{n-1}) := (\varphi(t_0), \varphi'(t_0), \cdots, \varphi^{(n-1)}(t_0))$ and let $C_1, \cdots, C_n \in \mathbb{R}$ be the solution to $\begin{pmatrix} \varphi_1(t_0) & \cdots, & \varphi_n(t_0) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(t_0) & & \varphi_n^{(n-1)}(t_0) \end{pmatrix}\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix}$. Note that the system above has a unique solution since $W(\varphi_1, \cdots, \varphi_n)(t_0) \neq 0$.

Claim: $\varphi(t) = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t)$. Since $\varphi(t)$, $C_1\varphi_1(t) + \cdots + C_n\varphi_n(t)$ are both solutions to [c] satisfying the same initial condition. Therefore, by Thm 4.1, the solution is unique, so the claim follows.

//.

Def 4.5: A collection of solutions $\{\varphi_1, \cdots, \varphi_n\}$ to [c] is called a fundamental set of [C] if $W(\varphi_1, \cdots, \varphi_n)(t) \neq 0$, for some t in the interval of interest. ▨

§4.1.1. Linear independence of Functions. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A collection of vectors $\{v_1, \cdots, v_n\}$ is called linearly dependent if there exist constants $(c_1, \cdots, c_n) \neq (0, \cdots, 0)$, such that $c_1 v_1 + \cdots + c_n v_n = 0$. If no such $c_1, \cdots, c_n$ exists, $\{v_1, \cdots, v_n\}$ is called linearly independent.

Let $V = \{$all $(n-1)$-times differentiable functions defined on an open interval $I\}$. V is a vector space. Given $\{f_1, \cdots, f_n\} \subseteq V$. Suppose that $c_1 f_1(t) + \cdots + c_n f_n(t) = 0$, for all $t \in I$. $\Rightarrow$ $c_1 f_1^{(k)} + \cdots + c_n f_n^{(k)}(t) = 0$, for all $t \in I$ and $k = 1, \cdots, n-1$. $\Rightarrow$ $\begin{pmatrix} f_1(t), & \cdots, & f_n(t) \\ \vdots & & \\ f_1^{(n-1)}(t) & & f_n^{(n-1)}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, for all $t \in I$.

If there exists a $t_0 \in I$ such that $\det \begin{pmatrix} f_1(t_0), & \cdots, & f_n(t_0) \\ \vdots & & \vdots \\ f_1^{(n-1)}(t_0) & & f_n^{(n-1)}(t_0) \end{pmatrix} \neq 0$, then $c_1 = \cdots = c_n = 0$ and $\{f_1, \cdots, f_n\}$ are linearly independent.

Therefore, a collection of solutions $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of [C] if and only if $\{\varphi_1, \cdots, \varphi_n\}$ are linearly independent. ▨

Lecture VIII. (5月27日)

I. ※※

§4.1.2: The Homogeneous equations - Reduction of orders. Suppose that $y=\varphi_1(t)$ is a solution to [c]. Now, we look for a function v so that $y=v(t)\varphi_1(t)$ is also a solution to [c]. The derivativ of v satisfies an $(n-1)$-th order homogeneous ordinary differential equati

Example 4.6: Suppose we are given $y=\varphi_1(t)=e^t$ as a solution to $(2-t)y''' + (2t-3)y'' - ty' + y = 0$ for $t<2$. [※] Suppose that $y=v(t)e^t$ is also a solution to [※]. $\Rightarrow (2-t)(v'''e^t + 3v''e^t + 3v'e^t + ve^t) + (2t-3)(v''e^t + 2v'e^t + ve^t) - t(v'e^t + ve^t) + ve^t = 0 \Rightarrow$ v satisfies $(2-t)v''' + (3-t)v'' = 0$ Let $v'' = u$. $\Rightarrow (2-t)u' + (3-t)u = 0 \Rightarrow u = C_1(2-t)e^{-t}$. $\Rightarrow v(t) = C_3 + C_2 t + C_1 e^{-t} - C_1(t+1)e^{-t} = C_3 + C_2 t - C_1 t e^{-t}$. We can check that a fundamental set of [※] is $\{e^t, te^{-t}, t\}$.

//

§4.1.3: The nonhomogeneous equations: Let $y=Y_1(t)$ and $y=Y_2(t)$ be solutions to [a], then $Y_1(t)-Y_2(t)$ is a solution to [c]. If $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of [c], then $Y_1(t) - Y_2(t) = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t)$

Lecture VIII.

II.**

Thm 4.7: The general solution of the nonhomogeneous equation [a] can be written in the form $y = \varphi(t) = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t) + Y(t)$, where $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of [c], $C_1, \cdots, C_n$ are arbitrary constants and $Y(t)$ is a particular solution to [a]. ∎

In general, in order to solve [a], we follow the procedure belw:

① Find fundamental set $\{\varphi_1, \cdots, \varphi_n\}$ of [c]

② Find a particular solution $Y(t)$ of [a]

③ Apply Thm 4.7.

§4.2: Homogeneous equations with constant coefficients. Consider n-th order linear homogeneous O.D.E with constant coefficients:

$$L[y] := y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \cdots \quad [e]$$

where a_j, $j = 0, 1, \cdots, n-1$, are constants. The characteristic equation of [e] is given by $\underbrace{r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0}_{[f]}$. Let $r_1, \cdots, r_n$ be solutions to [f]. Then [e] can be written as $\left(\frac{\partial}{\partial t} - r_1\right)\cdots\left(\frac{\partial}{\partial t} - r_n\right)y = 0$.

Lecture VIII.

III. **

① If $\{r_1, \cdots, r_n\}$ are distinct real roots, then $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \cdots + C_n e^{r_n t}$. [*I]

pf: Let $z_1 = (\frac{\partial}{\partial t} - r_2) \cdots (\frac{\partial}{\partial t} - r_n) y$. Then, $z_1' - r_1 z_1 = 0 \Rightarrow z_1(t) = C_1 e^{r_1 t}$.

Let $z_2 = (\frac{\partial}{\partial t} - r_3) \cdots (\frac{\partial}{\partial t} - r_n) y$. Then, $z_2' - r_2 z_2 = z_1$. By using the method of integrating factors, we find that $z_2(t) = \frac{C_1}{r_1 - r_2} e^{r_1 t} + C_2 e^{r_2 t}$.

Repeating the process, we conclude [*I].

② If $\{r_1, \cdots, r_n\}$ are distinct, and $r_{2k-1} = a_k + i b_k$, $r_{2k} = a_k - i b_k$, $k = 1, \cdots, \ell$,

$a_k \in \mathbb{R}$, $b_k \in \mathbb{R}$, $b_k \neq 0$, $k = 1, \cdots, \ell$, and $\{r_{2\ell+1}, \cdots, r_n\}$ are real.

Then $y(t) = C_1 e^{a_1 t} \cos b_1 t + C_2 e^{a_1 t} \sin b_1 t + \cdots + C_{2\ell-1} e^{a_\ell t} \cos b_\ell t + C_{2\ell} e^{a_\ell t} \sin b_\ell t$

$+ C_{2\ell+1} e^{r_{2\ell+1} t} + \cdots + C_n e^{r_n t}$. ⋯ [*II]

pf: Assume $r_1 = a_1 + i b_1$, $r_2 = a_1 - i b_1$ $\Rightarrow z_1' - r_1 z_1 = 0 \Rightarrow z_1 = C_1 e^{a_1 t}(\cos b_1 t + i \sin b_1 t)$

$\Rightarrow z_2 = C_1 e^{at} \cos bt + C_2 e^{at} \sin bt$, for some complex constants C_1, C_2.

③ If the characteristic equation of [e] has repeat roots. We group the roots in a way that $r_1 = r_2 = \cdots = r_\ell$ and so on. Then, $\frac{\partial}{\partial t}(e^{-r_2 t} z_2) = \cdots$

$e^{-r_2 t} z_1 = C_1 \Rightarrow z_2(t) = (C_1 t + C_2) e^{r_1 t}$. If $r_1 = r_2 = a_1 + i b_1$, $a_1 \in \mathbb{R}$, $b_1 \in \mathbb{R}$, $b_1 \cdots$

Then, $z_2(t) = (c_1 t + c_2) e^{(a+bi)t} = (c_1 t + c_2) e^{at}(\cos bt + i\sin bt)$, c_1, c_2 are complex constants. (i) Suppose that $r_1 = r_2 = r_3 = r$. Let $z_3 = (\frac{\partial}{\partial t} - r_4)\cdots(\frac{\partial}{\partial t} - r_n)y$.

$\Rightarrow z_3' - r z_3 = (c_1 t + c_2)e^{rt} \Rightarrow z_3(t) = (\frac{c_1}{2}t^2 + c_2 t + c_3)e^{rt}$.

(ii) Suppose that $r_1 = r_2 = r$, $r_3 \neq r_2$. Let $z_3 = (\frac{\partial}{\partial t} - r_4)\cdots(\frac{\partial}{\partial t} - r_n)y$.

$\Rightarrow z_3' - r_3 z_3 = (c_1 t + c_2)e^{rt} \Rightarrow z_3(t) = (\hat{c}_1 t + \hat{c}_2)e^{rt} + c_3 e^{r_3 t}$.

$\hat{c}_1 e^{rt} + r(\hat{c}_1 t + \hat{c}_2)e^{rt} - r_3(\hat{c}_1 t + \hat{c}_2)e^{rt} = (c_1 t + c_2)e^{rt}$ $\quad r\hat{c}_1 - r_3\hat{c}_1 = c_1$

Summing up, we get that if r_j's are roots of the characteristic equation of [e] with multiplicity n_j ($n_1 + n_2 + \cdots + n_k = n$), then the general solution to [e] is $y(t) = \sum_{j=1}^{k} p_j(t)e^{r_j t}$, where $p_j(t) = c_{j,0} + c_{j,1}t + \cdots + c_{j,n_j-1}t^{n_j-1}$, $c_{j,0}, c_{j,1}, \dots, c_{j,n_j-1}$ are constants, $j = 1, 2, \dots, k$. If $r_j = a + ib$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $b \neq 0$, then $e^{r_j t} = e^{at}(\cos bt + i\sin bt)$. Assume $r_1 = a + ib$, $r_2 = a - ib$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $b \neq 0$. Then, $n_1 = n_2$. $p_1(t)e^{r_1 t} + p_2(t)e^{r_2 t} = \hat{p}_1(t)e^{at}\cos bt + \hat{p}_2(t)e^{at}\sin bt$, $\hat{p}_1(t) = \hat{c}_{1,0} + \hat{c}_{1,1}t + \cdots + \hat{c}_{1,n_1-1}t^{n_1-1}$, $\hat{p}_2(t) = \hat{c}_{2,0} + \hat{c}_{2,1}t + \cdots + \hat{c}_{2,n_1-1}t^{n_2-1}$. In other words if r_j are real roots of [f] with multiplicity n_j and $a_k \pm ib_k$ are complex roots of [f] with multiplicity m_k ($\sum_j n_j + 2\sum_k m_k = n$), then the general solution to [e] is $y(t) = \sum_j p_j(t)e^{r_j t} + \sum_k e^{a_k t}(q_k^1(t)\cos b_k t + q_k^2(t)\sin b_k t)$ where $p_j(t)$'s are some polynomials of degree $n_j - 1$ and q_k^1, q_k^2's are some polynomials of degree $m_k - 1$.

I. ※※.

Example 4.8 : Find the general solution of $y^{(4)}+y'''-7y''-y'+6y=0$.

The roots of the characteristic equation $r^4+r^3-7r^2-r+6=0$ are $r=\pm 1$, $r=2$, $r=-3$. Thus, the general solution to the O.D.E. is $y=C_1e^{t}+C_2e^{-t}+C_3e^{2t}+C_4e^{-3t}$. If we look for a solution to the O.D.E. satisfying the initial conditions $y(0)=1$, $y'(0)=0$, $y''(0)=-1$, $y'''(0)=-1$, then
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \Rightarrow C_1=\frac{11}{8},\ C_2=\frac{5}{12},\ C_3=-\frac{2}{3},\ C_4=-\frac{1}{8}.$$
▨

Example 4.9 : Find the general solution of $y^{(4)}-y=0$. Also find the solution that satisfies the initial condition $y(0)=\frac{7}{2}$, $y'(0)=-4$, $y''(0)=\frac{5}{2}$, $y'''(0)=-2$. The roots of the characteristic equation $r^4-1=0$ are $r=\pm 1$, $r=\pm i$. The general solution to the O.D.E. above is $y=C_1e^{t}+C_2e^{-t}+C_3\cos t+C_4\sin t$. To satisfy the initial condition, $C_1,\cdots,C_4$ have to satisfy
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}=\begin{pmatrix} \frac{7}{2} \\ -4 \\ \frac{5}{2} \\ -2 \end{pmatrix} \Rightarrow C_1=0,\ C_2=3,\ C_3=\frac{1}{2},\ C_4=-1.$$
▨

Example 4.10 : Find the general solution of $y^{(4)}+y=0$. The roots of the characteristic equation are $r=\pm\left(\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i\right)$. The general solution of the O.D.E. above is $y=e^{\frac{\sqrt{2}}{2}t}\left(C_1\cos\frac{\sqrt{2}}{2}t+C_2\sin\frac{\sqrt{2}}{2}t\right)+e^{-\frac{\sqrt{2}}{2}t}\left(C_3\cos\frac{\sqrt{2}}{2}t+C_4\sin\frac{\sqrt{2}}{2}t\right)$.
▨

§4.4: The method of variation of parameters. To solve a non-homogeneous O.D.E. $L[y] := y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = g(t)$, [a] we often apply the method of variation of parameters to find a particular solution. Suppose that $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of the homogeneous equation $y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = 0$ ⋯ [b]. Assume that a particular solution can be written as $y = Y(t) = u_1(t)\varphi_1(t) + \cdots + u_n(t)\varphi_n(t)$. Assume that $u_1, \cdots, u_n$ satisfy $u_1'\varphi_1 + \cdots + u_n'\varphi_n = 0$, $u_1'\varphi_1' + \cdots + u_n'\varphi_n' = \cdots$ $u_1'\varphi_1^{(n-2)} + \cdots + u_n'\varphi_n^{(n-2)} = 0$. Then, $Y' = u_1\varphi_1' + \cdots + u_n\varphi_n'$, $Y'' = u_1\varphi_1'' + \cdots + u_n\varphi_n''$, $\cdots$, $Y^{(n-1)} = u_1\varphi_1^{(n-1)} + \cdots + u_n\varphi_n^{(n-1)}$, $Y^{(n)} = u_1'\varphi_1^{(n-1)} + \cdots + u_n'\varphi_n^{(n-1)} + u_1\varphi_1^{(n)} + \cdots + u_n\varphi_n^{(n)}$.

Since $Y(t)$ is assumed to be a particular solution to [a], we have $u_1'\varphi_1^{(n-1)} + \cdots + u_n'\varphi_n^{(n-1)} = g(t)$. Therefore, $u_1, u_2, \cdots, u_n$ satisfy the following

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & & \varphi_n' \\ \vdots & & & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & & \varphi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{pmatrix}.$$

Let W_m denote the Wronskian of $\{\varphi_1, \cdots, \varphi_{m-1}, \varphi_{m+1}, \cdots, \varphi_n\}$. That is, $W_m = \det\begin{pmatrix} \varphi_1 & \cdots & \varphi_{m-1} & \varphi_{m+1} & \cdots & \varphi_n \\ \varphi_1' & & \varphi_{m-1}' & \varphi_{m+1}' & & \varphi_n' \\ \varphi_1^{(n-2)} & & \varphi_{m-1}^{(n-2)} & \varphi_{m+1}^{(n-2)} & & \varphi_n^{(n-2)} \end{pmatrix}$. Then,

$$u_k' = (-1)^{n+k}\frac{W_k g}{W(\varphi_1, \cdots, \varphi_n)}, \quad k = 1, 2, \cdots, n. \Rightarrow Y(t) = \sum_{j=1}^{n}(-1)^{n+j}\varphi_j(t)\int_{t_0}^{t}\frac{W_j(s)g(s)}{W(\varphi_1, \cdots, \varphi_n)(s)}\,ds.$$

[b].

Lecture IX.

III. ※※

Example 4.11: Find the general solution to $y''' - y'' - y' + y = g(t)$ ⋯ (i).

Note that the roots of the characteristic equation $r^3 - r^2 - r + 1 = 0$ of the homogeneous equation $y''' - y'' - y' + y = 0$ ⋯ (ii) are $r = 1$ (double) and $r = -1$. Thus $\{e^t, te^t, e^{-t}\}$ is a fundamental set of (ii). Let $\varphi_1(t) = e^t$, $\varphi_2(t) = te^t$, $\varphi_3(t) = e^{-t}$. Then, $W(\varphi_1, \varphi_2, \varphi_3)(t) = \det\begin{pmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{pmatrix}$ $= 4e^t$. $W_1(t) = -2t - 1$, $W_2(t) = -2$, $W_3(t) = e^{2t}$. Therefore, a particular solution is $Y(t) = \sum_{j=1}^{3} (-1)^{3+j} \varphi_j(t) \int_0^t \frac{W_j(s) g(s)}{W(\varphi_1, \varphi_2, \varphi_3)(s)} ds = e^t \int_0^t \frac{(-2s-1) g(s)}{4e^s} ds$ $- te^t \int_0^t \frac{-2 g(s)}{4e^s} ds$. The general solution to (i) is $y = C_1 e^t + C_2 te^t + C_3 e^{-t} + Y(t)$.

$\boxtimes$

§7: System of First order linear equation: The study of system of first order linear O.D.E. plays an important role in science, applied math.

Example 7.1: Let $p: [0, +\infty) \to \mathbb{R}^+$ denote the population of certain species.
(掠食者-獵物方程)

The Lotka-Volterra equation or the predator-prey equation:

$$\begin{cases} p' = rp - \sigma pq \\ q' = \beta q + \delta pq \end{cases}$$ is a system of O.D.E.

p: number of prey (rabbits), q: number of some predator (foxes)

$\gamma, \sigma, \beta, \delta$ are real parameters describing the interaction of the two species.

▨

Example 7.2 : Suppose that we are considering a scalar n-th order O.D.E $y^{(n)}(t) = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t))$. Let $x_1(t) = y(t)$, $x_2(t) = y'(t), \cdots, x_n(t) = y^{(n-1)}(t)$. Then, $(x_1, \cdots, x_n)$ satisfies $[*]\begin{cases} x_1'(t) = x_2(t) \\ \vdots \\ x_{n-1}'(t) = x_n(t) \\ x_n'(t) = f(t, x_1, \cdots, x_n) \end{cases}$

Let $X = (x_1, \cdots, x_n)^T$, $F(t, x) = (x_2, x_3, \cdots, x_n, f(t, x_1, \cdots, x_n))^T$. Then $[*]$ can also be written as $x'(t) = F(t, x(t))$.

▨

Def 7.3 : A system of O.D.E is an equation of the form $\underset{[a]}{x'(t) = F(t, x(t))}$, where $F = (F_1, \cdots, F_n)^T$, $F_j : I_0 \times I_1 \times \cdots \times I_n \to \mathbb{R}$ is a function, $j = 1, 2, \cdots, n$, $x(t) = (x_1(t), \cdots, x_n(t))^T$, $x_j(t) : I_0 \to \mathbb{R}$ is a differentiable function, $j = 1, 2, \cdots, n$, $I_0, I_1, \cdots, I_n$, are open intervals of $\mathbb{R}$.

▨

Def 7.4 : The system of O.D.E. [a] is said to be linear if F is of the form $F(t, x) = P(t)x + g(t)$, where $P(t) = (P_{j,k}(t))_{j,k=1}^{n}$

V. ** Lecture IX.

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is a matrix. [a] is said to be homogeneous if $g(t)=0$. ▨

Example 7.5: Consider the second order O.D.E. $y''-2y'-2y=\sin t$. Let $x_1(t)=y(t)$ and $x_2(t)=y'(t)$. Then $x'(t)=\begin{pmatrix}0&1\\2&2\end{pmatrix}x(t)+\begin{pmatrix}0\\\sin t\end{pmatrix}$. More precisely, $\begin{pmatrix}x_1'(t)\\x_2'(t)\end{pmatrix}=\begin{pmatrix}0&1\\2&2\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}+\begin{pmatrix}0\\\sin t\end{pmatrix}$. The second order linear O.D.E. corresponds to a system of first order linear O.D.E. ▨

Example 7.6: The O.D.E. $x'=\begin{pmatrix}1&1\\4&1\end{pmatrix}x$ is a system of first order linear homogeneous O.D.E. To solve the O.D.E. $x'=\begin{pmatrix}1&1\\4&1\end{pmatrix}x$ with initial condition $x(t_0)=\begin{pmatrix}a\\b\end{pmatrix}$ means that we have to find $x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$ such that $\begin{pmatrix}x_1'\\x_2'\end{pmatrix}=\begin{pmatrix}1&1\\4&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\left(\begin{cases}x_1'=x_1+x_2\\x_2'=4x_1+x_2\end{cases}\right)$ with $x_1(t_0)=a$, $x_2(t_0)=b$. ▨

§7.2: Basic theory of system of first order O.D.E.

Thm 7.7: Let $x_0=(x_{1,0},x_{2,0},\cdots,x_{n,0})$ be a point in $\mathbb{R}^n$, $V\subset\mathbb{R}^n$ be an open set containing x_0. Let $F: I\times V\to\mathbb{R}^n$ be a real-valued function of t and x such that $F=(F_1,\cdots,F_n)$ and the partial

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derivative $\frac{\partial F_j}{\partial x_k}$ is continuous in $I \times V$, for all $j, k = 1, 2, \cdots, n$, where I is an open interval of $\mathbb{R}$. Fix $t_0 \in I$. Then in some interval $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \subset I$, where $\varepsilon > 0$, there exists a unique solution $X(t)$ to the initial value problem $X' = F(t, X)$, $X(t_0) = X_0$. $\cdots$ [b]

Moreover, if [b] is linear and $V = \mathbb{R}^n$, the solution exists throughout the interval I.

$\blacksquare$

Corollary 7.8: Let $(y_0, y_1, \cdots, y_{n-1})$ be a point in $\mathbb{R}^n$, $V \subset \mathbb{R}^n$ be an open set containing $X_0 \in V$. Let $f: I \times V \to \mathbb{R}$ be a function such that f is real valued and f, $\frac{\partial f}{\partial x_j}$, $j = 1, \cdots, n$, are continuous in $I \times V$, where I is an open set of $\mathbb{R}$. Fix $t_0 \in I$. Then in some interval $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \subset I$, where $\varepsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem $y^{(n)}(t) = f(t, y, y', \cdots, y^{(n-1)})$, $y(t_0) = y_0$, $y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}$. In particular, the solution $y(t)$ is n-times continuously differentable in $(t_0 - \varepsilon, t_0 + \varepsilon)$.

$\blacksquare$

Lecture IX.

pf: Consider $F: I \times V \to \mathbb{R}^n$, $F(t, x_1, \cdots, x_n) = (x_2, \cdots, x_n, f(t, x_1, \cdots, x_n))$.

By Thm 7.7, there exists $x(t) = (x_1(t), \cdots, x_n(t))$ defined on $(t_0 - \varepsilon, t_0 + \varepsilon)$, for some $\varepsilon > 0$, such that $x(t_0) = (y_0, y_1, \cdots, y_n)$ and $x'(t) = (x_1'(t), \cdots, x_n'(t)) = F(t, x(t)) = (x_2(t), \cdots, x_n(t), f(t, x_1(t), \cdots, x_n(t)))$.

Let $x_1(t) = y(t)$. Then, $y'(t) = x_2(t), \cdots, y^{(n-1)}(t) = x_n(t)$ and $y^{(n)}(t) = f(t, y(t), \cdots, y^{(n-1)}(t))$. The theorem follows.

$\blacksquare$

We will prove Thm 7.7 later.

$\blacksquare$

Lecture X. (3月30日)

I.**

DATE

Let $P(t)$ be a t-dependent $n\times n$ matrix. We write $M_{n\times n}$ denote the space of $n\times n$ real matrices. Let $P = P(t) : I \to M_{n\times n}$ be a matrix valued function, where I is an open interval of $\mathbb{R}$.

Consider the linear system $x'(t) = P(t)x(t)$, $x(t) = (x_1(t), \cdots, x_n(t))^T$. ⋯ [a]

Thm 7.9 : If the vectors φ_1 and φ_2 are solutions of [a]. Then the linear combination $c_1\varphi_1 + c_2\varphi_2$ is also a solution to [a] for any constants c_1 and c_2.

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Given n t-dependent vectors $\varphi_j(t) : I \to \mathbb{R}^n$, $j=1,\cdots,n$. We say that $\{\varphi_1(t), \cdots, \varphi_n(t)\}$ is linear independent if there is no $(c_1,\cdots,c_n) \neq (0,\cdots,0)$, such that $c_1\varphi_1(t) + \cdots + c_n\varphi_n(t) = 0$, for all $t \in I$.

Thm 7.10 : If the vector functions $\hat{\varphi}_j(t) : I \to \mathbb{R}^n$, $j=1,\cdots,n$, are linearly independent solutions to [a]. Then for each solution $x = \varphi(t)$ to [a], there exist a unique vector $(c_1,\cdots,c_n) \in \mathbb{R}^n$ such that $\varphi(t) = c_1\hat{\varphi}_1(t) + \cdots + c_n\hat{\varphi}_n(t)$.

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II. **

Lecture X.

DATE

pf: Let $e_j = (0, \cdots, 0, \underset{j\text{-th}}{1}, 0, \cdots, 0)$, $j = 1, 2, \cdots, n$. Fix $t_0 \in I$. For each $j = 1, 2, \cdots, n$, by Theorem 7.7, there exists a unique solution $\varphi_j(t)$ to [a] such that $\varphi_j(t_0) = e_j$.

The set $\{\varphi_1, \cdots, \varphi_n\}$ are linear independent. If not, there exists non-zero vectors $(c_1, \cdots, c_n)$ such that $c_1\varphi_1(t) + \cdots + c_n\varphi_n(t) = 0$, for all $t \in I$. Take $t = t_0$, we get $c_1 = \cdots = c_n = 0$. We get contradiction.

We note that $\{\varphi_1, \cdots, \varphi_n\}$ is a fundamental set of [a] since every solution $\underset{(x_1(t), \cdots, x_n(t))^T}{x(t)}$ to [a] can be uniquely expressed by $x(t) = x_1(t_0)\varphi_1(t) + x_2(t_0)\varphi_2(t) + \cdots + x_n(t_0)\varphi_n(t)$. In fact, $x(t)$ and $x_1(t_0)\varphi_1(t) + \cdots + x_n(t_0)\varphi_n(t)$ are both solutions to [a] satisfying the initial data $x(t_0) = (x_1(t_0), \cdots, x_n(t_0))^T$. By Thm 7.7, $x(t) = x_1(t_0)\varphi_1(t) + \cdots + x_n(t_0)\varphi_n(t)$. Now, $\text{span}(\hat{\varphi}_1, \cdots, \hat{\varphi}_n) \subseteq \text{span}(\varphi_1, \cdots, \varphi_n)$. Since $\{\hat{\varphi}_1, \cdots, \hat{\varphi}_n\}$ are linearly independent, $\dim \text{span}(\hat{\varphi}_1, \cdots, \hat{\varphi}_n) = n$. Since $\dim \text{span}(\varphi_1, \cdots, \varphi_n) = n$, we get $\text{span}(\hat{\varphi}_1, \cdots, \hat{\varphi}_n) = \text{span}(\varphi_1, \cdots, \varphi_n)$. $\boxed{///}$

Def 7.11: Let $\{\varphi_1(t), \cdots, \varphi_n(t)\}$ be a collection of solutions to [a]. $\{\varphi_1(t), \cdots, \varphi_n(t)\}$ is called a fundamental set of [a] if $\{\varphi_1(t), \cdots, \varphi_n(t)\}$ are linearly independent, the matrix $\Psi(t) := (\varphi_1(t), \cdots, \varphi_n(t))$ is called the fundamental matrix of [a] and $\varphi(t) = c_1\varphi_1(t) + \cdots + c_n\varphi_n(t)$ is called the general solution of [a].

Thm 7.12: If $\varphi_1, \cdots, \varphi_n$ are solutions to [a], then $\det(\varphi_1(t), \cdots, \varphi_n(t))$ is either identically zero or else never vanishes.

pf: Let $W(t) := \det(\varphi_1, \cdots, \varphi_n)$, $P = (p_{j,k})_{j,k=1}^{n}$, $\varphi_j = (\varphi_j^{(1)}, \cdots, \varphi_j^{(n)})^T$, $j = 1, \cdots, n$. Since φ_k is a solution to [a], we have $\varphi_k^{(j)\prime} = \sum_{l=1}^{n} p_{j,l}\varphi_k^{(l)}$, $j = 1, \cdots, n$.

$$\det\begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(j-1)} & \varphi_2^{(j-1)} & \cdots & \varphi_n^{(j-1)} \\ \varphi_1^{(j)\prime} & \varphi_2^{(j)\prime} & \cdots & \varphi_n^{(j)\prime} \\ \varphi_1^{(j+1)} & \varphi_2^{(j+1)} & \cdots & \varphi_n^{(j+1)} \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{pmatrix} = \det\begin{pmatrix} \varphi_1^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & & \vdots \\ \varphi_1^{(j-1)} & \cdots & \varphi_n^{(j-1)} \\ \sum_{l=1}^{n} p_{j,l}\varphi_1^{(l)} & \cdots & \sum_{l=1}^{n} p_{j,l}\varphi_n^{(l)} \\ \varphi_1^{(j+1)} & \cdots & \varphi_n^{(j+1)} \\ \vdots & & \vdots \\ \varphi_1^{(n)} & \cdots & \varphi_n^{(n)} \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_1^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & & \vdots \\ \varphi_1^{(j-1)} & \cdots & \varphi_n^{(j-1)} \\ p_{jj}\varphi_1^{(j)} & \cdots & p_{jj}\varphi_n^{(j)} \\ \varphi_1^{(j+1)} & \cdots & \varphi_n^{(j+1)} \\ \vdots & & \vdots \\ \varphi_1^{(n)} & \cdots & \varphi_n^{(n)} \end{pmatrix} = p_{jj}W.$$

Therefore,

$$\frac{\partial W}{\partial t} = \det\begin{pmatrix} \varphi_1^{(1)\prime} & \varphi_2^{(1)\prime} & \cdots & \varphi_n^{(1)\prime} \\ \varphi_1^{(2)} & \varphi_2^{(2)} & \cdots & \varphi_n^{(2)} \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n)} & \varphi_2^{(n)} & \cdots & \varphi_n^{(n)} \end{pmatrix} + \begin{pmatrix} \varphi_1^{(1)} & \cdots & \varphi_n^{(1)} \\ \varphi_1^{(2)\prime} & \cdots & \varphi_n^{(2)\prime} \\ \varphi_1^{(3)} & \cdots & \varphi_n^{(3)} \\ \vdots & & \vdots \\ \varphi_1^{(n)} & \cdots & \varphi_n^{(n)} \end{pmatrix} + \cdots + \begin{pmatrix} \varphi_1^{(1)} & \cdots & \varphi_n^{(1)} \\ \varphi_1^{(2)} & \cdots & \varphi_n^{(2)} \\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \\ \varphi_1^{(n)\prime} & \cdots & \varphi_n^{(n)\prime} \end{pmatrix} = (p_{11} + \cdots + p_{nn})W = (\operatorname{tr} P)W.$$

$\Rightarrow W(t) = \exp(\int_{t_0}^{t} tr(P)(s)ds)\, W(t_0)$ which implies that W is identically zero or else never vanishes.

$\boxed{////}$

Def 7.13: If $\varphi_1, \varphi_2, \cdots, \varphi_n$ are n solutions to [a]. The determinant $W(\varphi_1, \cdots, \varphi_n)(t) \equiv \det(\varphi_1, \varphi_2, \cdots, \varphi_n)$ is called the Wronskian of $\{\varphi_1, \cdots, \varphi_n\}$.

$\boxed{////}$

Thm 7.14: Let $u, v: I \to \mathbb{R}^n$ be real valued functions. If $x(t) = u(t) + iv(t)$ is a solution to [a], so are u and v.

$\boxed{////}$

pf: Since $x(t) = u(t) + iv(t)$ is a solution to [a], $x'(t) - P(t)x(t) = 0. \Rightarrow u'(t) + iv'(t) - P(t)(u(t) + iv(t)) = 0 = u'(t) - P(t)u(t) + i(v'(t) - P(t)v(t)) = 0. \Rightarrow u'(t) - P(t)u(t) = 0$ and $v'(t) - P(t)v(t) = 0$.

$\boxed{////}$

§7.3: Homogeneous linear systems with constant coefficients

In this section, we consider the equation $x'(t) = Ax(t)$, $\cdots$ [a]

where $A = (a_{j,k})_{j,k=1}^{n}$ is a constant $\underbrace{n \times n}_{\text{real}}$ matrix.

§7.3.1: The case that A has distinct real eigenvalues. Suppose that A has distinct real eigenvalues $\lambda_1, \cdots, \lambda_n$ with corresponding eigenvectors $v_1, \cdots, v_n$. That is, $Av_j = \lambda_j v_j$, $j = 1, \cdots, n$. Let $P = (v_1, \cdots, v_n)$, $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. Then, $A = P\Lambda P^{-1} \Rightarrow x'(t) = P\Lambda P^{-1}x(t) \Rightarrow P^{-1}x'(t) = \Lambda P^{-1}x(t)$. Let $y(t) = P^{-1}x(t)$. Then, $y'(t) = \Lambda y(t)$. Write $y(t) = (y_1(t), \cdots, y_n(t))^T$. Then, $\begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} \Rightarrow y_j'(t) = \lambda_j y_j(t)$, $j = 1, \cdots, n$. If $y(t_0) = y_0 = (y_{01}, \cdots, y_{0n})^T$ is given, we obtain

$$y(t) = \begin{pmatrix} e^{\lambda_1(t-t_0)} y_{01} \\ \vdots \\ e^{\lambda_n(t-t_0)} y_{0n} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1(t-t_0)} & & \\ & \ddots & \\ & & e^{\lambda_n(t-t_0)} \end{pmatrix} y_0.$$

Thus the solution of [a] with initial data $x(t_0) = x_0$ ($y_0 = P^{-1}x_0$) can be written as

$$x(t) = Py(t) = P \begin{pmatrix} e^{\lambda_1(t-t_0)} & & \\ & \ddots & \\ & & e^{\lambda_n(t-t_0)} \end{pmatrix} P^{-1} x_0 \cdots \text{(i)}$$

Def 7.15: Let $B \in M_{n\times n}$. The exponential of an $n \times n$ matrix B

is given by $e^{B} := I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots = \sum_{k=0}^{+\infty} \frac{1}{k!}B^k$.

From $(t\Lambda)^k = \begin{pmatrix}(\lambda_1 t)^k & & \\ & \ddots & \\ & & (\lambda_n t)^k\end{pmatrix}$, we find that $e^{t\Lambda} = \begin{pmatrix}\sum_{k=0}^{+\infty}\frac{1}{k!}(\lambda_1 t)^k & & \\ & \ddots & \\ & & \sum_{k=0}^{+\infty}\frac{1}{k!}(\lambda_n t)^k\end{pmatrix}$

$= \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix}$. (i) implies that $x(t) = P e^{(t-t_0)\Lambda} P^{-1} x_0$. Moreover,

$$x(t) = (v_1, \dots, v_n)\begin{pmatrix} e^{\lambda_1(t-t_0)} & & \\ & \ddots & \\ & & e^{\lambda_n(t-t_0)}\end{pmatrix}\begin{pmatrix}y_{01}\\ \vdots \\ y_{0n}\end{pmatrix} = y_{01}e^{\lambda_1(t-t_0)}v_1 + \cdots + y_{0n}e^{\lambda_n(t-t_0)}v_n.$$

$\Rightarrow$ The solution of [a] is a linear combination of vectors

$\{e^{\lambda_1(t-t_0)}v_1, \dots, e^{\lambda_n(t-t_0)}v_n\}$.

§7.3.2: The case when A has distinct complex eigenvalues.

Example 7.16: Find a fundamental set of real-valued solution of the system $x' = \begin{pmatrix}-\frac{1}{2} & 1\\ -1 & -\frac{1}{2}\end{pmatrix}x$. ...[*] We first diagonalize the matrix $A = \begin{pmatrix}-\frac{1}{2} & 1\\ -1 & -\frac{1}{2}\end{pmatrix}$ and find that $\begin{pmatrix}-\frac{1}{2} & 1\\ -1 & -\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$

$\begin{pmatrix}-\frac{1}{2}+i & 0\\ 0 & -\frac{1}{2}-i\end{pmatrix}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}^{-1} \Rightarrow \begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}^{-1} = -\frac{1}{2i}\begin{pmatrix}-i & -1\\ -i & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}, & -\frac{i}{2}\\ \frac{1}{2}, & \frac{i}{2}\end{pmatrix}$.

$$\Rightarrow \begin{pmatrix}\frac{1}{2} & -\frac{i}{2}\\ \frac{1}{2} & \frac{i}{2}\end{pmatrix}\begin{pmatrix}x_1'(t)\\ x_2'(t)\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}+i & 0\\ 0 & -\frac{1}{2}-i\end{pmatrix}\overbrace{\begin{pmatrix}\frac{1}{2}, & -\frac{i}{2}\\ \frac{1}{2}, & \frac{i}{2}\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}}^{=\binom{y_1}{y_2}} \Rightarrow \begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}c_1 e^{(-\frac{1}{2}+i)t}\\ c_2 e^{(-\frac{1}{2}-i)t}\end{pmatrix},$$

$\Rightarrow$ $\phi_1(t) = e^{(-\frac{1}{2}+i)t}\begin{pmatrix}1\\ i\end{pmatrix} = e^{-\frac{t}{2}}(\cos t + i\sin t)\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}e^{-\frac{t}{2}}\cos t\\ -e^{-\frac{t}{2}}\sin t\end{pmatrix} + i\begin{pmatrix}e^{-\frac{t}{2}}\sin t\\ e^{-\frac{t}{2}}\cos t\end{pmatrix}$,

$\phi_2(t) = e^{(-\frac{1}{2}-i)t}\begin{pmatrix}1\\ -i\end{pmatrix} = e^{-\frac{t}{2}}(\cos t - i\sin t)\begin{pmatrix}1\\ -i\end{pmatrix} = \begin{pmatrix}e^{-\frac{t}{2}}\cos t\\ -e^{-\frac{t}{2}}\sin t\end{pmatrix} - i\begin{pmatrix}e^{-\frac{t}{2}}\sin t\\ e^{-\frac{t}{2}}\cos t\end{pmatrix}$

are fundamental set of [*].

By Thm 7.14, $\varphi_1(t) = \begin{pmatrix} e^{-\frac{t}{2}}\cos t \\ -e^{-\frac{t}{2}}\sin t \end{pmatrix}$, $\varphi_2(t) = \begin{pmatrix} e^{-\frac{t}{2}}\sin t \\ e^{-\frac{t}{2}}\cos t \end{pmatrix}$ are also solutions to [a]. To see the linear indendence of φ_1 and φ_2, we note that the Wronskian of φ_1 and φ_2 is $W(\varphi_1, \varphi_2)(t) = \det\begin{pmatrix} e^{-\frac{t}{2}}\cos t, & e^{-\frac{t}{2}}\sin t \\ -e^{-\frac{t}{2}}\sin t, & e^{-\frac{t}{2}}\cos t \end{pmatrix} = e^{-t} \neq 0.$ $\Rightarrow \{\varphi_1, \varphi_2\}$ is a fundamental set of [*]. ///

Suppose that A is an $n \times n$ matrix which has $2k$ distinct complex eigenvalues denoted by $\lambda_1 + i\mu_1, \lambda_1 - i\mu_1, \cdots, \lambda_k + i\mu_k, \lambda_k - i\mu_k$, $\lambda_j \in \mathbb{R}, \mu_j \in \mathbb{R}$, $\mu_j \neq 0$, $j = 1, 2, \cdots, k$, and $n-2k$ distinct real eigenvalues $\lambda_{2k+1}, \cdots, \lambda_n$, with corresponding eigenvectors $\underbrace{u_+^{(1)}, u_-^{(1)}, \cdots, u_+^{(k)}, u_-^{(k)}, u_{2k+1}, \cdots, u_n}_{\text{linear independent over } \mathbb{C}}$, where $u_+^{(j)} = \overline{u_-^{(j)}} = a_j + i b_j$, a_j, b_j real vectors, $j = 1, 2, \cdots, k$

The general solution is of the form: $x(t) = \hat{C}_+^{(1)} u_+^{(1)} e^{(\lambda_1 + i\mu_1)t} + \hat{C}_-^{(1)} u_-^{(1)} e^{(\lambda_1 - i\mu_1)t} + \cdots + \hat{C}_+^{(k)} u_+^{(k)} e^{(\lambda_k + i\mu_k)t} + \hat{C}_-^{(k)} u_-^{(k)} e^{(\lambda_k - i\mu_k)t} + C_{2k+1} u_{2k+1} e^{\lambda_{2k+1} t} + \cdots + C_n u_n e^{\lambda_n t}$.

$u_+^{(1)} e^{(\lambda_1 + i\mu_1)t} = (a_1 + i b_1) e^{\lambda_1 t}(\cos\mu_1 t + i \sin\mu_1 t)$

$= (a_1 e^{\lambda_1 t}\cos\mu_1 t - b_1 e^{\lambda_1 t}\sin\mu_1 t) + i(b_1 e^{\lambda_1 t}\cos\mu_1 t + a_1 e^{\lambda_1 t}\sin\mu_1 t)$.

By Thm 7.14, $a_1 e^{\lambda_1 t}\cos\mu_1 t - b_1 e^{\lambda_1 t}\sin\mu_1 t$ and $b_1 e^{\lambda_1 t}\cos\mu_1 t + a_1 e^{\lambda_1 t}\sin\mu_1 t$